tt*-geometry and pluriharmonic maps

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Abstract

In this paper we use the real differential geometric definition of a metric (an unimodular oriented metric) tt^*-bundle of Cortés and the author [CS] to define a map Φ from the space of metric (unimodular oriented metric) tt^*-bundles of rank r over a complex manifold M to the space of pluriharmonic maps from M to $GL(r)/O(p,q)$ (respectively $SL(r)/SO(p,q)$), where (p,q) is the signature of the metric. In the sequel the image of the map Φ is characterized. It follows, that in signature $(r,0)$ the image of Φ is the whole space of pluriharmonic maps. This generalizes a result of Dubrovin [D].

Keywords: tt^*-geometry and tt^*-bundles, pluriharmonic maps, pseudo-Riemannian symmetric spaces.

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Contents

1 Introduction

tt^*-geometry is a geometry, which has its origin in physics. Around 1990 physicist began to study topological-field-theories and their moduli-spaces, in particular N=2-super-symmetric-field-theories. A special geometric structure called topological-anti-topolgical fusion was found and studied (see for example [CV] and [D]). A definition of tt^*-geometry on abstract vector bundles was formulated in [Her] and [Sch1]. The former tt^*-geometries are included in this version by choosing TM^c respectively $T^{1,0}M$ as the bundle in the abstract version. Mathematically this geometry can be considered as a generalization of variations of Hodge-structures (VHS), as it was done in a paper of Hertling [Her]. From his results follows, that a special Kähler-manifold gives a tt^*-bundle. A definition in terms of real differential geometry was given in [CS] and used to give another proof of this result not using the methods of VHS. A further interesting class of solutions are harmonic bundles first introduced by Simpson [Sim]. These solutions are considered in [Her] and [Sch2, Sch4].

A result of Dubrovin [D] associates to every tt^*-geometry with positive definite metric a pluriharmonic map to $GL(r)/O(r)$ where r is the dimension of the base-manifold and vice-versa to every such map a tt^*-geometry. This result was proven by the author in his 'Diplomarbeit' [Sch1] for the case of a tt^*-geometry on an abstract vector bundle and is presented here in a more general context. The explicit form of this map in the special Kähler case, which implies its pluriharmonicity, was given in [CS]. In this context indefinite metrics can occur. This is the motivativation to generalize the above result to the case of tt^*-bundles carrying indefinite metrics. In [Sch4] we applied the above result to harmonic bundles with hermitian metric of arbitrary signature and obtained a generalization of the correspondence between harmonic bundles over a compact Kähler manifold X of complex dimension n and harmonic maps from X to $GL(n,\mathbb{C})/U(n)$.

May we illustrate now the main results: In theorem 2 we show, that a metric tt^*-bundle with a metric of signature (p,q) over a complex manifold (M,J) gives rise to a pluriharmonic map f from M to $GL(r)/O(p,q)$ being admissible in the following sense

Definition 1 *Let (M,J) be a complex manifold and G/K a locally Riemannian symmetric space with associated Cartan-decomposition $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$. A map $f:(M,J)\to G/K$ is said to be* admissible, *if the linear extension of its differential maps $T^{1,0}_xM$ (respectively $T^{0,1}_xM$) to an Abelian subspace of $\mathfrak{p}^c$ for all $x\in M$.*

Conversely an admissible pluriharmonic map f from M to $GL(r)/O(p,q)$ gives rise to a metric tt^*-bundle as is shown in theorem 3. In other words we could say, that our construction defines a map Φ from the space of metric tt^*-bundles of rank r over a complex manifold (M,J) to the space of pluriharmonic maps from M to $GL(r)/O(p,q)$. The image of the map Φ is characterized to be the admissi ble pluriharmonic maps from M to $GL(r)/O(p,q)$. The case of a metric tt^*-bundle of rank r with metric of signature $(r,0)$ follows from this theorem, since in this case the pluriharmonic are shown to be admissible using a result of Sampson [Sam]. It remains the question, if these all these pluriharmonic maps are admissi ble or if there are some counter-examples, which we do not know yet. The descripted results are also proven for unimodular oriented metric tt^*-bundles. Here the target space of the pluriharmonic maps is $SL(r)/SO(p,q)$.
We hope this approach enables a broader readership to understand this result relating physical/algebro-geometrical objects with well-known differential-geometric objects.
The author wants to thank his advisor V. Cortés for the motivated support of his work, C. Hertling for the discussions and B. Dubrovin for the hints related to their papers. Moreover, he wants to thank his parents having enabled all leading to this work and M.-A. Lawn for herself.

2 tt*-bundles

For the convenience of the reader we recall the definition of a tt^*-bundle given in [CS]:

Definition 2 *A* tt*-bundle (E,D,S) *over a complex manifold* (M,J) *is a real vector bundle* $E \to M$ *endowed with a connection* D *and a section* $S \in \Gamma(T^*M \otimes \mathrm{End}\, E)$ *satisfying the* tt*-equation

$$R^\theta = 0 \quad \textit{for all} \quad \theta \in \mathbb{R}\,, \tag{2.1}$$

where R^θ *is the curvature tensor of the connection* D^θ *defined by*

$$D^\theta_X := D_X + \cos(\theta) S_X + \sin(\theta) S_{JX} \quad \textit{for all} \quad X \in TM\,. \tag{2.2}$$

A metric tt*-bundle (E,D,S,g) *is a tt*-bundle* (E,D,S) *endowed with a possibly indefinite* D*-parallel fiber metric* g *such that* S *is* g*-symmetric, i.e. for all* $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \textit{for all} \qquad X,Y,Z \in T_pM\,. \tag{2.3}$$

An unimodular metric tt*-bundle (E,D,S,g) *is a metric tt*-bundle* (E,D,S,g) *such that* $\mathrm{tr}\, S_X = 0$ *for all* $X \in TM$*. An* oriented unimodular metric tt*-bundle (E,D,S,g,or) *is an unimodular metric tt*-bundle endowed with an orientation* or *of the bundle* E*.*

In the case of moduli spaces of topological quantum field theories [CV, D] and the moduli spaces of singularities [Her], the complexified tt^*-bundle $E^{\mathbb{C}}$ is identified with $T^{1,0}M$ and the metric g is positive definite. The case $E = TM$, and hence $E^{\mathbb{C}} = T^{1,0}M + T^{0,1}M$ includes special complex and special Kähler manifolds, as was proven in [CS] and follows from [Her].

Remark 1
1) If (E, D, S) is a tt-bundle then (E, D, S^θ) is a tt*-bundle for all $\theta \in \mathbb{R}$, where*

$$S^\theta := D^\theta - D = (\cos\theta)S + (\sin\theta)S_J\,. \tag{2.4}$$

The same remark applies to metric tt-bundles.*
2) Notice that an oriented unimodular metric tt-bundle (E, D, S, g, or) carries a canonical metric volume element $\nu \in \Gamma(\wedge^r E^*)$, $r = \mathrm{rk}\, E$, determined by g and or, which is D^θ parallel for all $\theta \in \mathbb{R}$.*

The following proposition characterizes tt^*-bundles (E, D, S) in form of explicit equations for D and S. These equations are important in the later calculations

Proposition 1 *Let E be a real vector bundle over a complex manifold (M, J) endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \mathrm{End}\, E)$.*
Then (E, D, S) is a tt-bundle if and only if D and S satisfy the following equations:*

$$R^D + S \wedge S = 0, \quad S \wedge S \textit{ is of type (1,1)}, \quad d^D S = 0 \quad \textit{and} \quad d^D S_J = 0. \tag{2.5}$$

Proof: As the attentive reader may observe, it is easier to show this proposition after complexifying TM. But since one idea of the paper was to formulate these results in real differential geometry, we give a real version of the proof.
To prove the proposition, we have to compute the curvature of D^θ.
Let $X, Y \in \Gamma(TM)$ arbitrary:

$$\begin{aligned} R^\theta_{X,Y} &= R^D_{X,Y} + [D_X, \cos(\theta)S_Y + \sin(\theta)S_{JY}] + [\cos(\theta)S_X + \sin(\theta)S_{JX}, D_Y] \\ &+ [\cos(\theta)S_X + \sin(\theta)S_{JX}, \cos(\theta)S_Y + \sin(\theta)S_{JY}] - \cos(\theta)S_{[X,Y]} - \sin(\theta)S_{J[X,Y]} \\ &= R^D_{X,Y} + \sin^2(\theta)[S_{JX}, S_{JY}] + \cos^2(\theta)[S_X, S_Y] + \cos(\theta)\sin(\theta)([S_X, S_{JY}] + [S_{JX}, S_Y]) \\ &+ \cos(\theta)([D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}) + \sin(\theta)([S_{JX}, D_Y] + [D_X, S_{JY}] - S_{J[X,Y]}). \end{aligned}$$

We now recall the Fourier-expansion of

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta) \quad \text{and} \quad \sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

to find

$$\begin{aligned} R^\theta_{X,Y} &= R^D_{X,Y} + \frac{1}{2}([S_X, S_Y] + [S_{JX}, S_{JY}]) + \cos(\theta)([D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}) \\ &+ \sin(\theta)([S_{JX}, D_Y] + [D_X, S_{JY}] - S_{J[X,Y]}) + \frac{1}{2}\cos(2\theta)([S_X, S_Y] - [S_{JX}, S_{JY}]) \\ &+ \frac{1}{2}\sin(2\theta)([S_X, S_{JY}] + [S_{JX}, S_Y])\,. \end{aligned}$$

Taking Fourier-coefficients yields

$$\begin{aligned} 0 &= R^D_{X,Y} + \frac{1}{2}([S_X, S_Y] + [S_{JX}, S_{JY}])\,, \\ 0 &= [S_X, S_Y] - [S_{JX}, S_{JY}], \quad 0 = [S_X, S_{JY}] + [S_{JX}, S_Y], \\ 0 &= [D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}, \quad 0 = [S_{JX}, D_Y] + [D_X, S_{JY}] - S_{J[X,Y]} \end{aligned}$$

and equivalently

$$R^D_{X,Y} + [S_X, S_Y] = 0,\ S \wedge S(X, Y) = [S_X, S_Y] = [S_{JX}, S_{JY}],\ d^D S = 0 \text{ and } d^D S_J = 0.$$

□

3 Pluriharmonic maps

In this section we recall the notion of pluriharmonic maps and explain some properties of pluriharmonic maps to $S(p,q) := GL(r)/O(p,q)$ where $O(p,q)$ is the pseudo-orthogonal group of signature (p,q) respectively $S^1(p,q) := SL(r)/SO(p,q)$, which are needed later to prove the main theorem.
In order to uniform the formulation of the paper we introduce the following notions:

$$\begin{aligned} &G_0(r) = GL(r),\ G_1(r) = SL(r),\ \mathfrak{g}_0 = \mathfrak{gl}(r),\ \mathfrak{g}_1 = \mathfrak{sl}(r),\\ &K_0(p,q) = O(p,q),\ K_1(p,q) = SO(p,q),\ \mathfrak{k}_0 = \mathfrak{k}_1 = \mathfrak{so}(p,q),\\ &S^0(p,q) = S(p,q). \end{aligned}$$

These objects are also written with an index $i \in \{0;1\}$.

Definition 3 *Let (M,J) be a complex manifold and (N,h) a pseudo-Riemannian manifold with Levi-Civita connection ∇^h, D a connection on M which satisfies*

$$D_{JY}X = JD_YX \tag{3.1}$$

for all vector fields which satisfy $\mathfrak{L}_XJ = 0$ (i.e. for which $X - iJX$ is holomorphic) and ∇ the connection on $T^*M \otimes f^*TN$ which is induced by D and ∇^h.
A map $f : M \to N$ is pluriharmonic if and only if it satisfies the equation

$$\nabla''\partial f = 0\,, \tag{3.2}$$

where $\partial f = df^{1,0} \in \Gamma(\bigwedge^{1,0} T^*M \otimes_{\mathbb{C}} (TN)^{\mathbb{C}})$ is the $(1,0)$-component of d^cf and ∇'' is the $(0,1)$-component of $\nabla = \nabla' + \nabla''$.
Equivalently one regards $\alpha = \nabla d\phi \in \Gamma(T^*M \otimes T^*M \otimes \phi^*TN)$.
Then ϕ is pluriharmonic if and only if

$$\alpha(X,Y) + \alpha(JX,JY) = 0$$

for all $X,Y \in TM$.

Remark 2

1. *Note, that f is pluriharmonic iff f restricted to every holomorphic curve is harmonic. In fact, this gives a definition of pluriharmonic maps, which is independent of the choosen connections. For a short discussion of this the reader is referred to [CS].*

2. *Any complex manifold (M,J) admits a torsion free complex connection D (Complex means $DJ = 0$.) and consequently a connection satisfying (3.1). In the rest of the paper, we want therefore suppose, that the connection on (M,J) is also torsion free.*

Let $Sym^i_{p,q}(\mathbb{R}^r)$ be the symmetric $r \times r$ matrices in $G_i(r)$ of signature (p,q). These define pseudo-scalar-products of same signature by $\langle \cdot,\cdot\rangle_A = \langle A\cdot,\cdot\rangle_{\mathbb{R}^r}$, where $\langle\cdot,\cdot\rangle_{\mathbb{R}^r}$ is the Euclidean scalar-product. The natural action of an element $g \in G_i(r)$ is given by $\langle g^{-1}\cdot, g^{-1}\cdot\rangle_A = \langle (g^{-1})^tAg^{-1}\cdot,\cdot\rangle_{\mathbb{R}^r}$. This gives us an action of $G_i(r)$ $A \mapsto (g^{-1})^tAg^{-1}$ on $Sym^i_{p,q}(\mathbb{R}^r)$ which we use to identify $Sym^i_{p,q}(\mathbb{R}^r)$ with $S^i(p,q)$ in the following

Proposition 2 *Let Ψ^i be the canonical map $\Psi^i : S^i(p,q) \tilde{\rightarrow} Sym^i_{p,q}(\mathbb{R}^r) \subset G_i(r)$ where $G_i(r)$ carries the pseudo-Riemannian Ad-invariant trace-form. Then Ψ^i is a totally-geodesic immersion and a map f from a complex manifold (M,J) to $S^i(p,q)$ is pluriharmonic, iff the map $\Psi^i \circ f : M \to G^i(r)$ is pluriharmonic.*

Proof: The proof is done by expressing the map Ψ^i in terms of the well-known Cartan-immersion. For further information see for example [Hel], [CE], [GHL], [KN].

1) First we study the identification $S^i(p,q) \tilde{\rightarrow} Sym^i_{p,q}(\mathbb{R}^r)$.
By Sylvesters theorem $G_i(r)$ operates on $Sym^i_{p,q}(\mathbb{R}^r)$ via
$$G_i(r) \times Sym^i_{p,q}(\mathbb{R}^r) \to Sym^i_{p,q}(\mathbb{R}^r), \ \ (g,B) \mapsto g \cdot B := (g^{-1})^t B g^{-1}.$$
The stabilisator of the point $I_{p,q} = \mathrm{diag}(\mathbb{1}_p, -\mathbb{1}_q)$ is $K_i(p,q)$ and the above action is transitive by Sylvesters theorem. Therefore by the orbit-stabilizer theorem (compare Gallot, Hulin, Lafontaine [GHL] 1.100) we obtain a diffeomorphism $\Psi^i : S^i(p,q) \tilde{\rightarrow} Sym^i_{p,q}(\mathbb{R}^r), \ \ g\, K_i(p,q) \mapsto g \cdot I_{p,q} = (g^{-1})^t I_{p,q} g^{-1}$.

2) We recall some results about symmetric spaces (see: [CE]). Let G be a Lie-group and $\sigma : G \to G$ a group-homomorphism with $\sigma^2 = Id_G$. Let K denote the subgroup $K = G^\sigma = \{g \,|\, \sigma(g) = g\}$. The Lie-algebra $\mathfrak{g}$ of G decomposes in $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with $d\sigma_{Id_G}(\mathfrak{h}) = \mathfrak{h}, \quad d\sigma_{Id_G}(\mathfrak{p}) = -\mathfrak{p}$. And we have the following information: The map $\phi : G/K \to G$ with $\phi : [gK] \mapsto g\sigma(g^{-1})$ defines a totally geodesic immersion called the Cartan-immersion.
We want to utilize this:
Therefore we define $\sigma : G_i(r) \to G_i(r), \ g \ \mapsto (g^{-1})^\dagger$ where $g^\dagger = I_{p,q} g^t I_{p,q}$ is the adjoint with respect to the pseudo-scalar product $\langle \cdot, \cdot \rangle_{I_{p,q}} = \langle \cdot, I_{p,q} \cdot \rangle_{\mathbb{R}^n}$.
σ is obviously a homomorphism and an involution with $G_i(r)^\sigma = K_i(p,q)$. By a direct calculation one gets $d\sigma_{Id_G} = -h^\dagger$ and hence
$$\begin{aligned} \mathfrak{h} &= \{h \in \mathfrak{gl}(r) \,|\, h^\dagger = -h\} = \mathfrak{o}(p,q) = \mathfrak{so}(p,q), \\ \mathfrak{p} &= \{h \in \mathfrak{gl}(r) \,|\, h^\dagger = h\} =: \mathrm{sym}^i(p,q). \end{aligned}$$
Thus we end up with
$$\phi : S^i(p,q) \ \to \ G_i(r), \tag{3.3}$$
$$g \ \mapsto \ g\sigma(g^{-1}) = gg^\dagger = gI_{p,q}g^t I_{p,q} = R_{I_{p,q}} \circ \Psi^i \circ \Lambda(g). \tag{3.4}$$
Here R_h is the right multiplication by h and Λ is the map $\Lambda : G_i \to G_i, h \mapsto (h^{-1})^t$. Both maps are isometries of the invariant metric. Hence Ψ^i is a totally-geodesic immersion.

3) Pluriharmonicity is independent of the connection satisfying (3.1) chosen on M. Therefore we can take it torsion free (see remark 2). We calculate the tensor
$$\nabla df(X,Y) = \nabla^N_X(df(Y)) - df(D_X Y).$$
for holomorphic vector fields X, Y. The (1,1)-part of the second term vanishes for holomorphic X, Y, since
$$D_X Y + D_{JX} JY = D_X Y + J D_{JX} Y = D_X Y + J^2 D_X Y = 0.$$

Hence we have only to regard the Levi-Civita-connections on G_i and $G_i/K_i = S^i(p,q)$. Let $X, Y \in \Gamma(TM)$ holomorphic and calculate:

$$\nabla^{G_i}_X d(\Psi^i \circ f)(Y) = \nabla^{G_i}_X d\Psi^i(df(Y)) = \nabla^{G_i}_X \Psi^i_*(df(Y)) = \Psi^i_*(\nabla^{G_i/K_i}_X df(Y)) + II(X,Y)$$

where II is the second fundamental-form which vanishes, as the immersion is totally geodesic. This implies with the notation $\alpha^{G_i} = \nabla^{G_i} d(\Psi^i \circ f)$ and $\alpha^{G_i/K_i} = \nabla^{G_i/K_i} df$

$$\begin{aligned}
\alpha^{G_i}(X,Y) + \alpha^{G_i}(JX,JY) &= \nabla^{G_i}_X d(\Psi^i \circ f)(Y) + \nabla^{G_i}_{JX} d(\Psi^i \circ f)(JY) \\
&= \Psi^i_* \left(\nabla^{G_i/K_i}_X df(Y) + \nabla^{G_i/K_i}_{JX} df(JY) \right) \\
&= \Psi^i_* \left(\alpha^{G_i/K_i}(X,Y) + \alpha^{G_i/K_i}(JX,JY) \right).
\end{aligned}$$

Since Ψ^i is an immersion, the left side is zero iff the right is and the proof is finished. $\square$

Remark 3 *(compare [CS])*
Above we have identified $G_i(r)/K_i(p,q)$ with $Sym^i_{p,q}(\mathbb{R}^r)$ via Ψ^i.
Let us choose $o = eK_i(p,q)$ as base point and suppose that Ψ^i is chosen to map o to $I = I_{p,q}$. By construction Ψ^i is $G_i(r)$-equivariant. We identify the tangent-space $T_S \mathrm{Sym^i}_{p,q}(\mathbb{R}^r)$ at $S \in \mathrm{Sym}^i_{p,q}(\mathbb{R}^r)$ with the (ambient) vector space of symmetric matrices:

$$T_S \mathrm{Sym^i}_{p,q}(\mathbb{R}^r) = \mathrm{Sym}^i(\mathbb{R}^r) := \{A \in \mathfrak{g}_i(r) | A^t = A\}\,. \tag{3.5}$$

For $\Psi^i(\tilde{S}) = S$, the tangent space $T_{\tilde{S}} S^i(p,q)$ is canonically identified with the vector space of S-symmetric matrices:

$$T_{\tilde{S}} S^i(p,q) = \mathrm{sym}^i(S) := \{A \in \mathfrak{g}_i(r) | AS = SA^t\}\,. \tag{3.6}$$

Note that $\mathrm{sym^i}(I_{p,q}) = \mathrm{sym^i}(p,q)$.

Proposition 3 *The differential of $\varphi^i := (\Psi^i)^{-1}$ at $S \in \mathrm{Sym}^i_{p,q}(\mathbb{R}^r)$ is given by*

$$\mathrm{Sym}^i(\mathbb{R}^r) \ni X \mapsto -\frac{1}{2} S^{-1} X \in S^{-1}\mathrm{Sym}^i(\mathbb{R}^r) = \mathrm{sym}^i(S)\,. \tag{3.7}$$

Using this proposition we relate now the differentials

$$df_x : T_x M \to \mathrm{Sym}^i(\mathbb{R}^r) \tag{3.8}$$

of a map $f : M \to \mathrm{Sym}^i_{p,q}(\mathbb{R}^r)$ at $x \in M$ and

$$d\tilde{f}_x : T_x M \to \mathrm{sym}^i(f(x)) \tag{3.9}$$

a map $\tilde{f} = \varphi \circ f : M \to S^i(p,q)$: $d\tilde{f}_x = d\varphi\, df_x = -\frac{1}{2} f(x)^{-1} df_x$.

One can interpret the 1-form $A = -2d\tilde{f} = f^{-1}df$ with values in $\mathfrak{g}_i(r)$ as connection form on the vector bundle $E = M \times \mathbb{R}^r$. We note, that the definition of A is the pure gauge. This means, that A is gauge-equivalent to $A' = 0$, as for $A' = 0$ one has $A = f^{-1}A'f + f^{-1}df = f^{-1}df$. The curvature vanishes, since it is independent of gauge. Thus we get:

Proposition 4 *Let $f : M \to G_i(r)$ be a C^∞-mapping and $A := f^{-1}df \; : TM \to \mathfrak{g}_i(r)$. Then the curvature of A vanishes, i.e. for $X, Y \in \Gamma(TM)$*

$$Y(A_X) - X(A_Y) = A_{[X,Y]} - [A_X, A_Y]. \tag{3.10}$$

In the next proposition we give the equations for pluriharmonic maps from a complex manifold to $G_i(r)$.

Proposition 5 *Let (M, J) be a complex manifold, $f : M \to G_i(r)$ a C^∞-map and A defined as in proposition 4.*
The pluriharmonicity of f is equivalent to the equation

$$Y(A_X) + \frac{1}{2}[A_Y, A_X] + JY(A_{JX}) + \frac{1}{2}[A_{JY}, A_{JX}] = 0, \tag{3.11}$$

for holomorphic $X, Y \in \Gamma(TM)$.

Proof: Again pluriharmonicity of f does not depend on the connection satisfying (3.1) on M. Hence the (1,1)-part of the second term of $\nabla df(X, Y)$ vanishes for holomorphic X, Y, as in the proof of proposition 2. Therefore we only have to regard the pulled back Levi-Civita connection ∇ on $G_i(r)$.
Let $X, Y \in \Gamma(TM)$. To find these equations we write $df(X)$ and $df(Y)$ that are sections in $f^*T\,G_i(r)$, as linear combination of left invariant vector fields $f^*\tilde{E}_{ij} = \tilde{E}_{ij} \circ f$, with $\tilde{E}_{ij}(g) = gE_{ij}$, $\forall g \in G_i(r)$ and a basis $E_{ij}, i, j = 1 \ldots r$ of $\mathfrak{g}_i(r)$.
In this notation we have

$$df(X) = \sum_{i,j} a_{ij}\, \tilde{E}_{ij} \circ f = \sum_{ij} a_{ij}\, fE_{ij} \text{ and } df(Y) = \sum_{i,j} b_{ij}\, \tilde{E}_{ij} \circ f = \sum_{ij} b_{ij}\, fE_{ij},$$

with functions a_{ij} and b_{ij} on M and further

$$A_X = f^{-1}df(X) = \sum_{ij} a_{ij}\, E_{ij} \text{ and } A_Y = f^{-1}df(Y) = \sum_{ij} b_{ij}\, E_{ij}.$$

With this information we compute

$$\begin{aligned}
(f^*\nabla)_Y df(X) &= (f^*\nabla)_Y \sum_{i,j} a_{ij}\, \tilde{E}_{ij} \circ f \\
&= \sum_{ij} Y(a_{ij})\, \tilde{E}_{ij} \circ f + \sum_{ij} a_{ij}\, (f^*\nabla)_Y \tilde{E}_{ij} \circ f \\
&= \sum_{ij} Y(a_{ij})\, \tilde{E}_{ij} \circ f + \sum_{ij} a_{ij}\, \nabla_{df(Y)} \tilde{E}_{ij} \circ f \\
&= \sum_{ij} Y(a_{ij})\, fE_{ij} + \sum_{abij} a_{ij}\, b_{ab}\, \underbrace{(\nabla_{\tilde{E}_{ab}} \tilde{E}_{ij}) \circ f}_{\frac{1}{2}f[E_{ab}, E_{ij}]} \\
&= f(Y(A_X) + \frac{1}{2}[A_Y, A_X]).
\end{aligned}$$

Therefore the pluriharmonicity is equivalent to the equation

$$Y(A_X) + \frac{1}{2}[A_Y, A_X] + JY(A_{JX}) + \frac{1}{2}[A_{JY}, A_{JX}] = 0$$

for holomorphic X, Y. $\square$

Suppose that N is a locally Riemannian symmetric space with universal cover G/K with non-compact semi-simple Lie group G, maximal compact subgroup K and associated Cartan-decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. In each point one identifies the tangent space of N with $\mathfrak{p}$. This is unique up to right action of K and left action of the fundamental group. All relevant structures are preserved by these actions. Therefore, given a $f : M \to N$, we can regard $df_x(T_x^{1,0}M)$, $x \in M$ as a subspace of $\mathfrak{p}^c$. For the 'complexified' sectional-curvature of N holds using the Killing-form b

$$b(R(X,Y)\bar{Y}, \bar{X}) = -b([X,Y], [\bar{Y}, \bar{X}]) \leq 0. \tag{3.12}$$

It is a well-known result of Sampson [Sam], that a harmonic map of a compact complex manifold to a locally symmetric space of non-compact type is pluriharmonic and that its differential sends $T^{1,0}M$ to an Abelian subspace of $\mathfrak{p}^c$. The second claim, that the image of $T^{1,0}M$ under the differential of a pluriharmonic map is Abelian is true on non-compact manifolds, too. To illustrate this, we are going to prove, that pluriharmonicity implies this property.

Theorem 1 *(compare [Sam]) Let (M, J) be a complex manifold and $N = G/K$ be a locally Riemannian symmetric space as above.*
Then the complex linear extended differential of a pluriharmonic map $f : M \to N$ maps for all $x \in M$ $T_x^{1,0}M$ (respectively $T_x^{0,1}M$) to an Abelian subspace of $\mathfrak{p}^c$.
On TM the differential of a pluriharmonic map $f : M \to N$ obeys the equation

$$[df_x(X), df_x(Y)] = [df_x(JX), df_x(JY)]$$

with $X, Y \in T_xM, x \in M$.

Proof: The strategy is to show the vanishing of the curvature.
Let $X, Y, Z, W \in \Gamma(T^{1,0}M)$ be holomorphic

$$\begin{aligned} R^N(f_*X, f_*Y)f_*\bar{Z} &= R^{f^*\nabla^N}(X,Y)f_*\bar{Z} \\ &= (f^*\nabla^N)_X(f^*\nabla^N)_Y f_*\bar{Z} - (f^*\nabla^N)_Y(f^*\nabla^N_X)f_*\bar{Z} - (f^*\nabla^N)_{[X,Y]}f_*\bar{Z} \end{aligned}$$

We remark now, that the pluriharmonic equation for holomorphic vector fields depends not on the connection chosen on the manifold M. Hence it reduces to the equation $(f^*\nabla^N)_X f_*\bar{Y} = 0$, which implies $R^N(f_*X, f_*Y)f_*\bar{Z} = 0$. From equation (3.12) we get $b([f_*X, f_*Y], [f_*\bar{Z}, f_*\bar{W}]) = 0$ and in the end $[f_*X, f_*Y] = 0$ for all X, Y.
Let $Z, W \in \Gamma(T^{1,0}M)$ be of the form $Z = X - iJX$ and $W = Y - iJY$ with $X, Y \in \Gamma(TM)$ and compute $[f_*Z, f_*W] = [f_*X, f_*Y] - [f_*JX, f_*JY] - i([f_*X, f_*JY] + [f_*JX, f_*Y])$. Hence we conclude $[df(X), df(Y)] = [df(JX), df(JY)]$. $\square$

Corollary 1 *Let (M, J) be a complex manifold, $f : M \to Sym^i_{r,0}(\mathbb{R}^r) \xhookrightarrow{\iota} G_i(r)$ a pluriharmonic map induced by a pluriharmonic map to $G_i(r)/K_i(r)$ and A defined as in proposition 4. If f is a pluriharmonic map, then the operators A satisfy for all $X, Y \in T_xM$, with $x \in M$, the equation $[A_X, A_Y] = [A_{JX}, A_{JY}]$.*

Proof: First, we apply theorem 1 to $A = -2d\tilde{f} : M \to G_1 = SL(r)$. This yields the corollary for $G_1 = SL(r)$.
For $S^0(p, q) = S(p, q)$ we have the de Rham-decomposition $S(p, q) = \mathbb{R} \times S^1(p, q)$, where $\mathbb{R}$ corresponds to the connected central subgroup $\mathbb{R}^{>0} = \{\lambda Id | \lambda > 0\} \subset G_0 = GL(r)$. Hence we have the decomposition of $\mathfrak{gl}(r) = \mathbb{R} \oplus \mathfrak{sl}(r)$, where the $\mathbb{R}$-factor is central. Therefore we are in the situation to apply the result for G_1. □

Remark 4 *Since the trace-form on $SL(r)$ is a multiple of the Killing-form and on $GL(r)$ it corresponds to the metric on the decomposition $S(p, q) = \mathbb{R} \times S^1(p, q)$, we can choose the trace-form as metric and obtain the same result as in theorem 1 and corollary 1.*

4 tt^*-geometry and pluriharmonic maps

In this section we are going to state and prove the main results. Like in section 3 one regards the mapping $A = f^{-1}df$ as a map $A : TM \to \mathfrak{g}_i(r)$
We now suppose, that the complex manifold (M, J) is simply connected. Using the same considerations as in [Sch1] the main theorems, theorem 2 and theorem 3, can be extended to non-simply connected manifolds by pulling back the metric tt^*-bundles to the unviversal cover of M. The according pluriharmonic maps have to be replaced by twisted pluriharmonic maps

Theorem 2 *Let (M, J) be a simply-connected complex manifold. Let $(E, D, S, g\,[, or]\,)$ be a metric [an unimodular oriented metric] tt^*-bundle where E has rank r and M dimension n.*
Then the representation of the metric g in a D^θ-flat frame of E $f : M \to Sym^i_{p,q}(\mathbb{R}^r)$ induces an admiseable pluriharmonic map $\tilde{f} : M \xrightarrow{f} Sym^i_{p,q}(\mathbb{R}^r) \xrightarrow{\tilde{\ }} S^i(p, q)$, where $S^i(p, q)$ carries the metric induced by the biinvariant pseudo-Riemannian trace-form on $\mathfrak{g}_i(r)$.
Let s' be another D^θ-flat frame. Then $s' = s \cdot U$ for a constant matrix and the pluriharmonic map associated to S' is $f' = U^t f U$.

Remark 5 *(see also [CS]) Before proving the theorem we make some remarks on the condition on $d\tilde{f}$. Let $x \in M$ and $\tilde{f}(x) = uo$. If $d\tilde{f}(T^{1,0}_x M)$ consist of commuting matrices, then $dL_u^{-1} d\tilde{f}(T^{1,0}_x M)$ is commutative, too. This follows from the fact, that*

$$dL_u \, : \, T_o S^i(p, q) \to T_{uo} S^i(p, q) = T_{\tilde{f}(x)} S^i(p, q)$$

equals

$$Ad_u \, : \, sym^i(p, q) = sym^i(I_{p,q}) \to sym^i(u \cdot I_{p,q}) = sym^i(\tilde{f}),$$

which preserves the Lie-bracket.

Proof: Using remark 1.1) it suffices to prove the case $\theta = \pi$.
We first consider a metric tt^*-bundle (E, D, S, g).
Let $s := (s_1, \dots, s_r)$ be a D^π-flat frame of E (i.e. $Ds = Ss$), f the matrix $g(s_k, s_l)$ and further S^s the matrix-valued one-form representing S in the frame s. For $X \in \Gamma(TM)$ we get:

$$\begin{aligned} X(f) &= Xg(s,s) = g(D_X s, s) + g(s, D_X s) \\ &= g(S_X s, s) + g(s, S_X s) \\ &= 2g(S_X s, s) = 2f \cdot S^s(X) = 2f \cdot S^s_X. \end{aligned}$$

Consequently $A_X = 2S^s_X$. We now prove the pluriharmonicity using

$$d^D S(X,Y) = D_X(S_Y) - D_Y(S_X) - S_{[X,Y]} = 0, \tag{4.1}$$

$$d^D S_J(X,Y) = D_X(S_{JY}) - D_Y(S_{JX}) - S_{J[X,Y]} = 0. \tag{4.2}$$

The equation (4.2) implies

$$\begin{aligned} 0 = d^D S_J(JX,Y) &= D_{JX}(S_{JY}) + \underbrace{D_Y(S_X)}_{\overset{(4.1)}{=} D_X(S_Y) - S_{[X,Y]}} - S_{J[JX,Y]} \\ &= D_{JX}(S_{JY}) + D_X(S_Y) - S_{[X,Y]} - S_{J[JX,Y]}. \end{aligned}$$

In local holomorphic coordinate fields X, Y on M we get in the frame s

$$JX(S^s_{JY}) + X(S^s_Y) + [S^s_X, S^s_Y] + [S^s_{JX}, S^s_{JY}] = 0.$$

Now $A = 2S^s$ gives equation (3.11) and proves the pluriharmonicity of f.
Using $A_X = 2S^s_X = -2d\tilde{f}(X)$, we find the property of the differential, as $S \wedge S$ is of type (1,1) using the tt^*-equations, see proposition 1.
The last statement is obvious.
In the case of an oriented unimodular metric tt^*-bundle (E, D, S, g, or) we can take the frame s to be oriented and of volume 1, with respect to the canonical D^θ-parallel- metric volume ν. Therefore the map f takes values in $Sym^1_{p,q}(\mathbb{R}^r)$ and the above arguments show the rest. □

Theorem 3 *Let (M, J) be a simply-connected complex manifold and put $E = M \times \mathbb{R}^r$. Then a pluriharmonic map $\tilde{f} : M \to S^i(p,q)$ give rise to a pluriharmonic map $f : M \xrightarrow{\tilde{f}} S^i(p,q) \tilde{\to} Sym^i_{p,q}(\mathbb{R}^r) \overset{\iota}{\hookrightarrow} G_i(r)$.*
If $\tilde{f}$ is admissible, then the map f induces a metric tt^*-bundle [an unimodular oriented metric tt^*-bundle] $(E, D = \partial + S, S = -d\tilde{f}, g = \langle f\cdot, \cdot\rangle_{\mathbb{R}^r} [, or])$ on M where ∂ is the canonical flat connection on E and or is the canonical orientation on E.

Remark 6 *We observe, that for Riemannian surfaces $M = \Sigma$ the condition on the differential holds, since $T^{1,0}\Sigma$ is one-dimensional.*

Proof:
Let $\tilde{f} : M \to S^i(p,q)$ be a pluriharmonic map. Then by proposition 2 we know, that $f : M \tilde{\to} Sym^i_{p,q}(\mathbb{R}) \overset{\iota}{\hookrightarrow} G_i(r)$ is pluriharmonic.
Since $E = M \times \mathbb{R}^r$, we can regard sections of E as r-tuples of $C^\infty(M,\mathbb{R})$-functions.
In the spirit of section 3 we regard the one form $A = -2d\tilde{f} = f^{-1}df$ with values in $\mathfrak{g}_i(r)$ as a connection on E. We remind, that the curvature of this connection vanishes (proposition 4).

a) First, we check the conditions on the metric:

Lemma 1 *The connection D is compatible with the metric g and S is symmetric with respect to g.*

Proof: This is a direct computation with $X \in \Gamma(TM)$ and $v, w \in \Gamma(E)$ using the relations $(*)$ $S = \frac{1}{2}f^{-1}df$, $(**)$ $df_x : T_xM \to T_{f(x)}\mathrm{Sym}^i_{p,q}(\mathbb{R}^r) = \mathrm{Sym}^i(\mathbb{R}^r)$ (compare remark 3) and $g = \langle f\cdot,\cdot\rangle_{\mathbb{R}^r} = \langle\cdot, f\cdot\rangle_{\mathbb{R}^r}$ which follows from $f : M \to \mathrm{Sym}^i_{p,q}(\mathbb{R}^r)$:

$$\begin{aligned} X(g(v,w)) &= X(\langle fv,w\rangle_{\mathbb{R}^r}) = \langle X(f)v,w\rangle_{\mathbb{R}^r} + \langle f(\partial_X v),w\rangle_{\mathbb{R}^r} + \langle fv,\partial_X w\rangle_{\mathbb{R}^r} \\ &\overset{(**)}{=} \frac{1}{2}\langle X(f)v,w\rangle_{\mathbb{R}^r} + \frac{1}{2}\langle v,X(f)w\rangle_{\mathbb{R}^r} + \langle f(\partial_X v),w\rangle_{\mathbb{R}^r} + \langle fv,\partial_X w\rangle_{\mathbb{R}^r} \\ &= \frac{1}{2}\langle f\cdot f^{-1}(X(f))v,w\rangle_{\mathbb{R}^r} + \frac{1}{2}\langle v,f\cdot f^{-1}(X(f))w\rangle_{\mathbb{R}^r} \\ &\quad + \langle f\partial_X v,w\rangle_{\mathbb{R}^r} + \langle fv,\partial_X w\rangle_{\mathbb{R}^r} \\ &\overset{(*),(**)}{=} g(X.v + S_X v,w) + g(v,X.w + S_X w) = g(D_X v,w) + g(v,D_X w). \end{aligned}$$

For $x \in M$ $d\tilde{f}_x$ takes by remark 3 values in $\mathrm{sym}^i(f(x))$. This shows that $S = -d\tilde{f}$ is symmetric with respect to $g = \langle f\cdot,\cdot\rangle_{\mathbb{R}^r}$. □

To finish the proof, we have to check the tt^*-equations. The second tt^*-equation

$$[S_X, S_Y] = [S_{JX}, S_{JY}] \tag{4.3}$$

for S follows from the assumption that the image of $T^{1,0}M$ under $d^c\tilde{f}$ is Abelian. In fact, this is equivalent to $[d\tilde{f}(JX), d\tilde{f}(JY)] = [d\tilde{f}(X), d\tilde{f}(Y)]\ \forall X, Y \in TM$.

$$\begin{aligned} d^D S(X,Y) &= [D_X, S_Y] - [D_Y, S_X] - S_{[X,Y]} \\ &= \partial_X(S_Y) - \partial_Y(S_X) + 2[S_X,S_Y] - S_{[X,Y]} = 0 \end{aligned}$$

is equivalent to the vanishing of the curvature of $A = 2S$ interpreted as a connection on E (see proposition 4).
Finally one has for holomorphic coordinate fields $X, Y \in \Gamma(TM)$

$$\begin{aligned} d^D S_J(JX,Y) &= [D_{JX}, S_{JY}] + [D_Y, S_X] = [\partial_{JX} + S_{JX}, S_{JY}] + [\partial_Y + S_Y, S_X] \\ &= \partial_{JX}(S_{JY}) + \partial_Y(S_X) + [S_{JX}, S_{JY}] - [S_X, S_Y] \\ &\overset{(4.3)}{=} \frac{1}{2}\left(\partial_{JX}(A_{JY}) + \partial_Y(A_X)\right) \overset{(3.10)}{=} \frac{1}{2}\left(\partial_{JX}(A_{JY}) + \partial_X(A_Y) + [A_X, A_Y]\right) \\ &\overset{(4.3)}{=} \frac{1}{2}\left(\partial_{JX}(A_{JY}) + \partial_X(A_Y) + \frac{1}{2}[A_X,A_Y] + \frac{1}{2}[A_{JX},A_{JY}]\right) \overset{(3.11)}{=} 0. \end{aligned}$$

This shows the vanishing of the tensor $d^D S_J$.
It remains to show the curvature equation for D. We observe, that $D+S=\partial+A$ and that A is flat, to find

$$0=R^{D+S}_{X,Y} \;=\; R^D_{X,Y}+d^D S(X,Y)+[S_X,S_Y] \overset{d^D S=0}{=} R^D_{X,Y}+[S_X,S_Y].$$

b) With the same proof as in part a) we get a metric tt^*-bundle. The orientation is given by the orientation of $E=M\times\mathbb{R}^r$.
It remains to check the condition on the trace of S. This property is clear, since in this case $d\tilde{f}_x$ takes values in $\mathrm{sym}^1(f(x))$ for all $x\in M$. □

We want to emphasize the last result in the positive definite case:

Theorem 4 *Let (M,J) be a simply-connected complex manifold and put $E=M\times\mathbb{R}^r$. Then a pluriharmonic map $\tilde{f}: M\to S^i(r,0)$ is admissible. Moreover, it induces a second pluriharmonic map $f: M \xrightarrow{\tilde{f}} S^i(r,0)\tilde{\to}Sym^i_{r,0}(\mathbb{R}^r) \overset{\iota}{\hookrightarrow} G_i(r)$ and a metric tt^*-bundle $(E,D=\partial+S,S=-d\tilde{f},g=\langle f\cdot,\cdot\rangle_{\mathbb{R}^r}\,[,or])$ on M where ∂ is the canonical flat connection on E and or is the canonical orientation of E.*

Proof: In the case of signature $(r,0)$ corollary 1 implies that for all $x\in M$ the image of $d\tilde{f}_x$ is Abelian and the differential of any pluriharmonic map $\tilde{f}: M\to S(r,0)$ is admissiable as required in theorem 3. □

In the situation of theorem 3 the two constructions are inverse in the following sense:

Proposition 6

1. *Let $(E,D,S,g\,[,or])$ be a metric [an unimodular oriented metric] tt^*-bundle on a complex manifold (M,J) and let $\tilde{f}$ be the associated pluriharmonic map constructed to a D^θ-flat frame s in theorem 2. Then $\tilde{f}$ is admissible and the metric [unimodular oriented metric] tt^*-bundle $(M\times\mathbb{R}^r,\tilde{D}=\partial+\tilde{S},\tilde{S},\tilde{g},[or])$ associated to $\tilde{f}$ in theorem 3 is the representation of $(E,D,S,g\,[,or])$ in the frame s.*

2. *Given a pluriharmonic map $\tilde{f}$ from a complex manifold (M,J) to $S^i(p,q)$, then one obtains via theorem 3 a metric [an unimodular oriented metric] tt^*-bundle $(M\times\mathbb{R}^r,D,S,g\,[,or])$. The pluriharmonic map associated to this metric tt^*-bundle is conjugated to the map $\tilde{f}$ by a constant matrix in $G_i(r)$.*

Proof: Using again remark 1.1) we can set $\theta=\pi$.

1. The maps f, $\tilde{f}$ and the metric $\tilde{g}=\langle f\cdot,\cdot\rangle_{\mathbb{R}^r}$ express the metric g in the frame s. In the computations of theorem 2 and with theorem 3 one finds $2\tilde{S}=A=f^{-1}df=2S^s$. From $0=D^\pi s=Ds-Ss$ we obtain that the connection D in the frame s is just $\partial+S^s=\partial+\frac{A}{2}=\partial+\tilde{S}=\tilde{D}$.

2. To find the pluriharmonic map associated to $(M\times\mathbb{R}^r,D,S,g\;[,or])$ we have to express the metric g in a D^π-flat frame s. But $D^\pi=\partial+\frac{A}{2}-\frac{A}{2}=\partial$. Hence we can take s as the standard-basis of $\mathbb{R}^r$ and we get f. Every other basis gives a conjugated result. □

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